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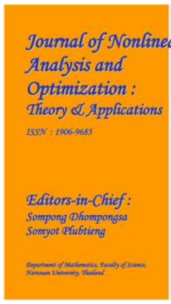

FINITE GROUPS CONSISTING OF JUST TWO SIZE INDEPENDENT GENERATING SETS

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Abstract:
In according to this work, A generating set S for a group G is independent if and only if ,for every s ϵS, the subgroup created by S$\backslash\{s\}$ is correctly contained in G. We characterize the structure of finite groups G such that the cardinalities of independent generating sets for G appear as exactly two integers.

Keywords:
Simple groups, generating sets, Engel elements, Soluble groups.

1- Introduction

The lowest number of generators of a finite group G is denoted by d(G) . A generating set S for a group G is independent (also known called irredundant) if

$$\langle S \setminus \{s\}\rangle < G \text{ for all s} \in S.$$

Let m(G) represent the largest independent generating set size for G. By Apisa and Klopsch, the finite groups with m(G) = d(G) are categorized.

1.1 Theorem:$(\boldsymbol{Apisa - Klopsch}, [\boldsymbol{Theorem\ 1.6}])$. G is soluble if d(G) = m(G). Additionally, either

- For a prime p, G/Frat(G) is an elementary abelian p-group; or
- G/Frat(G) = PQ, where Q is a nontrivial cyclic q-group for distinct primes p and q, and P is an elementary abelian p-group. Q acts faithfully on P via conjugation, and P (as a module for Q) is the direct sum of m (G)-1 isomorphic copies of a single simple Q-module.

Given this outcome, Apisa and Klopsch propose a straight forward "classification problem": identify all finite groups G that satisfy m(G)-d(G) ≤ c, given a nonnegative integer c. Glasby has lately drawn attention to the specific instance c = 1 (see[7,Problem 2.3]).

For every positive integer k with d(G) ≤k≤m(G), G contains an independent generating set of cardinality k, according to a nice result in universal algebra known as the Tarski irredundant basis theorem (see, for example, [3,Theorem 4.4]). Therefore, the condition m(G) –d(G) =1 is equivalent to the fact that there are only two possible cardinalities for an independent generating set of G.

Consider a finite group, G. Remember that the subgroup created by G's minimal normal subgroups is called the socle of G, or soc(G). Additionally, G is considered monolithic primitive if it has a single minimal normal subgroup, and the identity of G's Frattini subgroup Frat(G) is G. We establish the two primary outcomes listed below in this work.

1.2 Theorem: Assume that G is a finite group with m(G) = d(G) +1 and Frat(G) = 1.G is a monolithic primitive group and G/soc (G) is cyclic of prime power order if G is not soluble, as shown by d(G) = 2.Whiston and Sax[15] demonstrated that for any prime p that is not congruent to ± 1 modulo 8 or 10, m(PSL(2,p))=3. Specifically, we infer that there are an unlimited number of nonabelian simple groups G with m(G) = d(G)+1 since d(S) = 2 for every non abelian simple group. In Section 4, we additionally provide examples of non simple groups G such that m(G) = d(G)+1.

1.3 Theorem: Assume that G is a finite group with m(G) = d(G) +1 and Frat (G) = 1. In the event that G is soluble, one of the following happens:

(i) G ≅V⋊P, where V is an irreducible P-module that is not a p-group and P is a finite non cyclic p-group; in this instance, d(G) = d(P);
(ii) G ≅ V'⋊ H, where V is an irreducible faithful H-module, m(H) = 2, and either t = 1 or H is abelian; in this instance, d(G) = t+1;
(iii) Let $N_2/N_1 \leq$ Frat(G/N_2) and G /$N_2 \cong v^t \rtimes$ H be two normal subgroups of G such that $1 \leq N_1 \leq N_2$, N_1 is an abelian minimum normal subgroup of G; in this case, d(G) = t+1. Let V be an irreducible H-module and H be a nontrivial cyclic group of prime power order.
Examples of finite soluble groups G with m(G) = d(G) +1 for each of the three scenarios emerging from theorem 1.3 are provided in Section 4.

2 – Preliminary results

Let A be the unique minimal normal subgroup of monolithic primitive group L. Suppose that L^k is the k-fold direct product of L for any positive integer k. The subgroup L_k of L^k, defined by, is the crown-based power of L of size k.

$$L_1 \coloneqq \{(l_1, \dots, l_k) \in L^k | l_1 \equiv \cdots \equiv l_k \; mod \; A\}.$$

[4] establishes that for each finite group G, there is a homomorphic image L_k of G and a monolithic group L such that
(1) d$(L\,/socL) < d(G)$; and
(2) d$(L_k) = d(G)$.
This kind of group L_k is known as a generating crown-based power for G.
The explicit computation of d(d_k) in terms of k and the structure of L is discussed in [4]. Evaluating the maximal k for each monolithic group L such that L_k is a homomorphic image of G is a crucial step in determining d(G) from the behavior of the crown-based power homomorphic images of (G). The primary factors of G have an equivalency relation that gives birth to this number, k. Here we provide some information.
If groups G and A act on each other via automorphisms, then A is a G-group. Furthermore, if G does not stabilize any nontrivial proper subgroups of A, then A is irreducible. If there is a group isomorphism ∅:A →B such that ∅(g(a))=g(∅(a)) for all a ∈A and g ∈G, then two G-groups, A and B, are G-isomorphic. In accordance with [8], we declare that two irreducible G-groups, A and B, designated as A $\sim_G$B,are G-equivalent if an isomorphism $\boldsymbol{\Phi}$:A ⋊ G→B⋊G exists, which confines to a G-isomorphism ∅:A →B and generates the identity G ≅ AG/A→BG/B ≅G. Put another way, this implies that the diagram commutes.

$$\begin{array}{ccccccc} 1 \rightarrow & A & \rightarrow & A \rtimes G & \rightarrow & G & \rightarrow 1 \\ & \downarrow \emptyset & & \downarrow \boldsymbol{\Phi} & & \| & \rightarrow \mathbf{1} \\ 1 \rightarrow & B & \rightarrow & B \rtimes G & \rightarrow & G & \rightarrow 1 \end{array}$$

Observe that two G-isomorphic G-groups are G-equivalent ,and the converse holds if A and B are abelian.
Let A = X/Y is a Frattini chief factor if X/Y is contained in the Frattini subgroup of G/y ; this is equivalent to saying that A is abelian and there is no complement to A in G. The number $\delta_G(A)$ of non –Frattini chief factors that are G-equivalent to A, in any chief series of G, does not depend on the particular choice of such a series.
Now, we denote by $L_G(A)$ the monolithic primitive group associated to A, that is,

$$L_G(A) := \begin{cases} A \rtimes \left(G/C_G(A)\right) & if\ A\ is\ abelian, \\ G/C_G(A) & otherwise. \end{cases}$$

If A is a non-Frattini chief factor of G, then $L_G(A)$ is a homomorphic image of G. More precisely, there exists a normal subgroup N such that G/N ≅ $L_G(A)$ and soc $(G/N) \sim_G A$. We identify soc$\left(L_G(A)\right)$ with A, as G-groups.
Consider now all the normal subgroups N of G with the property that G/N ≅ $L_G(A)$ and soc $(G/N) \sim_G$ A. The intersection $R_G(A)$ of all these subgroups has the property that G/$R_G(A)$ is isomorphic to the crown –based power $\left(L_G(A)\right)_{\delta_G(A).}$The soce $I_G(A)/\,R_G(A) of\ G/R_G(A)$ is called the A-crown of G and it is a direct product of $\delta_G(A)$ minimal normal subgroups G-equivalent to A.

Note that if L is monolithic primitive and L_K is a homomorphic image of G for some $k \geq 1$, then L $\cong L_G(A)$ for some non-Frattini chief factor A of G and $k \leq \delta_G(A)$ for some non-Frattini chief factor A of G and $k \leq \delta_G(A)$. Furthermore, if $\left(L_G(A)\right)_k$ is a generating crown-based power, then so is $\left(L_G(A)\right)_{\delta_G(A)}$; in this case, we say that A is a generating chief factor for G.

For an irreducible G-module M, set

$$r_G(M) \coloneqq dim_{End_G(M)} \mathrm{M},$$
$$S_G(M) \coloneqq dim_{End_{G(M)}} H^1(G, M),$$
$$t_G(M) \coloneqq dim_{End_{G(M)}} H^1(G/C_G(M), M).$$

It can be seen that

$$S_G(M) = t_G(M) + \delta_G(M)$$

$(see\ for\ example$ [10, 1.2]). Now ,define

$$h_G(M) \coloneqq \begin{cases} \delta_G(M) & if\ M\ is\ a\ trivial\ G-module, \\ \left\lfloor \dfrac{S_G(M) - 1}{r_G(M)} \right\rfloor + 2 = \left\lfloor \dfrac{\delta_G(M) + t_G(M) - 1}{r_G(M)} \right\rfloor + 2 & otherwise\ . \end{cases}$$

By [2,Theorem A] , $t_G(M) < r_G(M)$ for any irreducible G-module M, and therefore

$$h_G(M) \leq \delta_G(M) + 1 \qquad (2.1)$$

The importance of $h_G(M)$ is clarified by the following proposition.

Proposition 2.1 [6,Proposition 2.1]. If there exists an abelian generating chief factor A of G, then d$d(G) = h_G(A)$.

When G admits a non abelian generating chief factor A, a relation between $\delta_G(A)$ and d(G) is provided by the following result.

Proposition 2.2. If d(G) ≥ 3 and there exists a nonabelian generating chief factor A of G, then

$$\delta_G(A) > \frac{|A|^{d(G)-1}}{2|C_{AutA}(L_G(A)/A)|} \geq \frac{|A|^{d(G)-2}}{2\ log_2|A|}\ .$$

Proof: Let A be a non-abelian generating main factor of G and assume that d(G) ≥3. Let $\emptyset_X$ (m) represent the number of ordered m-tuples $(x_1, \dots \dots\ x_n)$ of elements of X that generate X for a finite group X. Describe

$$\mathrm{L} := L_G(A),$$
$$\gamma := |C_{Aut\ A}(L/A)|,$$
$$\delta := \delta_G(A),$$
$$\mathrm{d} := \mathrm{d}(G).$$

In [4] ,it is proved that if m $\geq d(L), then$

$$\mathrm{d}(L_k) \leq m \text{ if and only if } \mathrm{k} \leq \frac{\emptyset_{L/A}(m)}{\emptyset_L(m)\gamma}\ . \qquad (2.2)$$

By the main result in [13],d$(L) = max\left(2, d(L/A)\right)$. Since A is a generating chief factor, from the definition, we have d$(L/A) < d\left(L_{\delta_G(A)}\right) = d(G)$. As 2 <d(G) , it follows d(L) <d(G). Now ,by applying (2.2) with k = $\delta_G(A)$ and m = d(G) -1, we deduce that

$$\delta_G(A) > \frac{\emptyset_{L/A}(d(G)-1)}{\emptyset_L(d(G)-1)\gamma} \qquad (2.3)$$

By [6,Corollary 1.2]

$$\frac{\emptyset_{L/A}(d(G)-1)}{\emptyset_L(d(G)-1)} \geq \frac{|A|^{d(G)-1}}{2}\ . \qquad (2.4)$$

Moreover, A $\cong S^n$,where n is a positive integer and S is a nonabelian simple group.

In the proof of Lemma 1 in [5] ,it is shown that

$$\gamma \leq n|S|^{n-1}|Aut\ (S)|.$$

Now, [9] shows that $|Out(S)| \leq log_2(|S|)$ and hence

$$\gamma \leq n|S|^n log_2(|S|) \leq |S|^n log_2(|S|^n) = |A| log_2(|A|). \qquad (2.5)$$

From (2.3) ,(2.4) and (2.5) ,we obtain

$$\delta_G(A) > \frac{\emptyset_{L/A}(d(G)-1)}{\emptyset_L(d(G)-1)\gamma} \geq \frac{|A|^{d(G)-1}}{2|A|LOG_2|A|} = \frac{|A|^{d(G)-2}}{2log_2|A|}.$$

Recall that m(G) is the largest cardinality of an independent generating set of G.

Theorem 2.3 [14,Theorem 1.3]. Consider a finite group, G. Then, for every primary series of G, m(G) = a+b, where a and b denote the number of non-Frattini and nonabelian components, respectively. Moreover, m(G) = a if G is soluble.

3- Proof of the main results

Given a finite group G, let d: = d(G) and let m: =m(G) . Suppose that m = d+1. Let A be a generating chief factor of G and let $\delta := \delta_G(A), L := L_G(A)$.

3.1 A is nonabelian: First ,suppose that $\delta \geq 2$. By Theorem 2.3 ,m $\geq 2\delta$ and therefore d $\geq 2\delta$ -1 $\geq$ 3. By Proposition 2.2,

$$\delta > \frac{|A|^{d-2}}{2log_2|A|} \geq \frac{|A|^{2\delta-3}}{2log_2|A|} \geq \frac{60^{2\delta-3}}{2log_2 60},$$

But this is never true.

Suppose now that $\delta = 1$. In this case, by the main theorem in [13], d = d(L) = max(2,d(L/A)) =2 and therefore m =3. Since L is an epimorphic image of G, we must have m(L)$\leq$ 3. However ,m(L)$\geq$ 3 by Corollary 2.4. Hence ,m(L) = m=3 and therefore it follows from [11,Lemma11] that G/Frat (G)$\cong L$.Finally, by Theorem 2.3, m(L) =3 implies m(L/A) ≤ 1, and this is possible only if L/A is a cyclic p-group .This concludes the proof of Theorem 1.2.

3.2 A is abelian: It follows from Proposition 2.1 and (2.1) that

$$\delta - 1 \leq m - 1 = d = h_G(A) \leq \delta + 1.$$

If d = $\delta - 1$, then m = δ and this is possible if G/Frat(G) $\cong A^{\delta}$. However, in this case, A would be a trivial G-module and therefore d = $h_G(A) = \delta = m$, which is a contradiction.

Now suppose that d= δ. By Theorem 2.3, G is soluble and contains only one non-Frattini chief factor which is not G-isomorphic to A. If A is noncentral in G, then G/Frat(G) $\cong L_\delta$ and L/A is a cyclic p-group . However, this implies $r_G(A) = 1, t_G(A)$ = 0 and d = $h_G(A) = \delta + 1$, which is a contradiction .If A is central, then G/Frat(G) $\cong V \rtimes P$,where P is a finite p-group, V is an irreducible P-module and d(P) =d. In particular, we obtain item (1) in Theorem 1.3.

Finally assume d = $\delta + 1$. Notice that in this case, L= A $\rtimes$ H ,where A is a faithful, nontrivial ,irreducible H-module, and

$$\text{m(H)} \leq m - \delta = \delta + 2 - \delta = 2.$$

In particular, by Corollary 2.4, H is soluble.

If m(H) =2, then G/Frat (G) $\cong L_\delta$. In particular ,we obtain item (2) in Theorem 1.3. If m(H) =1, then there exist two normal subgroups N_1 and N_2 of G such that $1 \leq N_1 \leq N_2, G/N_2 \cong L_\delta$,$N_2/N_1 \leq Frat(G/N_1)$ and $N_1/Frat(G)$ is an abelian minimal normal subgroup of G/Frat (G). As m(H) =1, H is cyclic of prime power order. In particular ,we obtain item (3) in Theorem 1.3.

4- Examples for Theorems 1.2 and 1.3

4.1 Monolithic groups: Examples for Theorem: 1.2. Let G be a monolithic primitive, S $\cong S_i$ for each 1 ≤ I ≤ n, and a nonabelian socle N = $S_1 \times ... \times S_n$. A study on the number μ(G) = m(G)-m (G/N) may be found in [12]. Using conjugation, the group G operates on the set $\{S_1, ... S_n\}$ of N's simple components. A group homomorphism G →Sym(n) is thus produced. Additionally, the subgroup X Aut S that results from N_G (S_1)'s conjugation action on the factor S_1 is a nearly simple group with socle S.

By [12,Proposition 4],$\mu(G) \geq \mu(X) = m(X) - m(X/S)$.Assume m(G) =3. Observe that by Theorems 1.1 and 1.2 ,G/N is cyclic of prime power order .If X = S, then

$$3 = \text{m(G)} = \text{m(G/N)} + \mu(G) \geq m(G/N) + \mu(X) = m(G/N) + m(S)$$
$$\geq m(G/N) + 3.$$

This implies that G/N =1 and G = S is a simple group. If X ≠ S,then G $\neq N$ and

$$3 = \text{m}(G) \geq m(G/N) + \mu(G) \geq 1 + \mu(X).$$

Furthermore, X/S is a prime power order nontrivial cyclic group, so

$$\text{m}(X) = m(X/S) + \mu(X) \leq 1 + \mu(X) \leq 1 + 2 = 3$$

By Corollary 2.4,m(X) =3.

The groups

$$\text{P}\sum L_2(9), M_{10}, Aut\big(PSL_2(7)\big)$$

Currently, to the best of the authors' knowledge, these are the only instances of nearly simple groups X with X ≠soc (X) and m(X) = 3. We think there are more situations like this, but the efficiency of our existing computer codes prevents us from conducting a thorough analysis.
Let S := $PSL_2(7)$ and H:= Aut $\left(PSL_2(7)\right)$, or let S := $PSL_2(9)$ and H ∈ $\{P\sum L_2\,(9)M_{10}\}$. Consider the wreath product W := H ≀Sym(n). Any element w ∈ W can be written as w = $\pi(a_1, \dots a_n)$, with $\pi \in Sym(n)$ and $a_i \in H$ for 1 $\leq i \leq n$. In particular , N = soc(W) = $S_1 \times \dots \times S_n = \{(s_1, \dots\dots s_n) | s_i \in S\}$.
4.2 –Soluble groups: Examples for Theorem 1.3: We provide three simple examples, but one can create more complex instances using the same concepts. Let C_n be the cyclic group of order n and S_n the symmetric group of degree n.
The group G: = $S_3 \times C_2^t = C_3 \colon C_2^{t+1}$ *with* $t \geq 1$ satisfies d(G) = t+1 and m(G) = t+2. This provides instances of groups that meet Theorem 1.3 item (1). M(G) =d(G)+1 is also satisfied by the groups G:= $(C_3^t \times C_2) \times C_2$ with C_2 acting on C_3^t via inversion and G: = S_4 =K :S_3, where K is the Klein subgroup of S_4. Groups satisfying item (2) in Theorem 1.3 are produced by these two cases, whereby H is abelian in the second case and m(H) = 2 in the first.
Let K be the Klein subgroup of S_4 as mentioned before and let G := K : $(S_3 \times C_2^{t-1})$. This gives examples of groups satisfying item (3) in Theorem 1.3.

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